Gibbs versus non-Gibbs distributions in money dynamics

Marco Patriarca [a], Anirban Chakraborti [b], and Kimmo Kaski [a]

[a] *Complex Systems Group, Laboratory of Computational Engineering, Helsinki University of Technology, P.O.Box 9203, 02015 HUT, Finland*

[b] *Department of Physics, Brookhaven National Laboratory, Upton, New York 11973, USA*

Abstract

We review a simple model of closed economy, where the economic agents make money transactions and a saving criterion is present. We observe the Gibbs distribution for zero saving propensity, and non-Gibbs distributions otherwise. While the exact solution in the case of zero saving propensity is already known to be given by the Gibbs distribution, here we provide the explicit analytical form of the equilibrium distribution for the general case of nonzero saving propensity. We verify it through comparison with numerical data and show that it can be cast in the form of a Poisson distribution.

Key words: Econophysics, money dynamics, Poisson distribution, Gibbs distribution
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1 Introduction

It is known that the higher end of the distribution of income $f(m)$ follows the Pareto law [1], $f(m) \propto m^{-1-\alpha}$, where m is the income (money) and the exponent α has a value in the interval 1 and 2 [2,3,4,5]. An explanation of the Pareto law, in terms of the laws regulating the system micro-dynamics, should take into account its basic constituents, i. e. the trading agents, as well as the criteria used to carry out the economic transactions. Several studies have been made to provide an explanation (see Ref. [6] for a brief summary and

Email addresses: marco@lce.hut.fi (Marco Patriarca), anirban@bnl.gov (Anirban Chakraborti), kimmo.kaski@hut.fi (Kimmo Kaski).

more references). In this respect, it is of general interest to study some simple systems of closed economy, which can be either solved exactly or simulated numerically, in order to investigate the relation between the micro-dynamics and the resulting macroscopic money distribution [7,8,9,10,11,12]. In this paper we consider the generalization of a simple model of money conserving economy, realized by introducing a criterion of saving in the transaction law, through the saving propensity λ. We study numerically its asymptotic money distribution as a function of the model parameters. We show that it is not a Gibbs distribution and, by direct comparison with numerical data, that the corresponding analytical solution has the form of a Poisson distribution.

2 Model

In the simple model considered [8], N agents can exchange money in pairs between themselves. For the sake of simplicity we assume that all the agents are initially assigned the same money amount m_0, despite this condition is not restrictive for the following results. Agents are then let to interact. At every "time step", a pair (i, j) is randomly chosen and the transaction takes place. During the transaction, the agent money amounts m_i and m_j undergo a variation, $m_i \to m'_i$ and $m_j \to m'_j$. Money is assumed to be conserved during the transaction, so that

$$m_i + m_j = m'_i + m'_j \ . \tag{1}$$

In this basic model, m'_i and m'_j are obtained through a random reassignment of the total money $(m_i + m_j)$,

$$\begin{aligned} m'_i &= \epsilon\,(m_i + m_j) \ , \\ m'_j &= (1-\epsilon)(m_i + m_j) \ , \end{aligned} \tag{2}$$

where ϵ is a random number, extracted from a uniform distribution in the interval $(0, 1)$. Notice that this model of dynamics, as well as its variations considered in the following, ensures that agents have no debts after the transaction, i. e. they are always left with a money amount $m \geq 0$. It can be shown that, merely as a consequence of the conservation law (1), the system relaxes toward an equilibrium state characterized by a Gibbs distribution [7,8,9],

$$f(m) = \beta \exp(-\beta m) \ , \tag{3}$$

where $\beta = 1/\langle m \rangle$ represents the inverse average money and $\langle m \rangle = \sum_i m_i/N \equiv m_0$. This means that after relaxation, the majority of the agents has a very small amount of money, while the number of richest agents – e. g. those with m larger than a given value m', as well as the fraction of the total money they own, exponentially decreases with m'. The Gibbs distribution (3) has been shown to represent a robust equilibrium state, reached independently of the initial conditions also in generalized models, such as those involving multi-agent transactions.

However, if a saving criterion is introduced [7,9], i.e. agents save a fraction λ – the saving propensity – of the money they have before the transaction is made, the shape of the equilibrium distribution changes dramatically. The conservation equation (1) still holds, but the money to be shared in a transaction between the i-th and the j-th agent is now $(1-\lambda)(m_i + m_j)$. Then Eqs. (2) are thus modified,

$$\begin{aligned} m_i' &= \lambda m_i + \epsilon(1-\lambda)(m_i + m_j) \ , \\ m_j' &= \lambda m_j + (1-\epsilon)(1-\lambda)(m_i + m_j) \ . \end{aligned} \tag{4}$$

These equations can also be rewritten in the following way,

$$\begin{aligned} m_i' &= m_i + \Delta m \ , \\ m_j' &= m_j - \Delta m \ , \\ \Delta m &= (1-\lambda)[\epsilon m_j - (1-\epsilon)m_i] \ , \end{aligned} \tag{5}$$

which clearly shows how money is conserved during the transaction.

We performed numerical simulations, for various values of λ, of a system with $N = 500$ agents. In each simulation a sufficient number of transactions, as far as 10^7, depending on the value of λ, was used in order to reach equilibrium. The final equilibrium distributions for a given λ, obtained by averaging over 1000 different runs, are shown in Fig. 1.

3 Fitting

The exact solution for the case $\lambda = 0$ is known to be given by the Gibbs distribution, Eq. (3). Here we give the corresponding exact solution for an generic value of λ, with $0 < \lambda < 1$. This solution was found by fitting the results of numerical simulations and it turns out to fit extremely well all data.

It is convenient to introduce the reduced variable

$$x = \frac{m}{\langle m \rangle} \ , \tag{6}$$

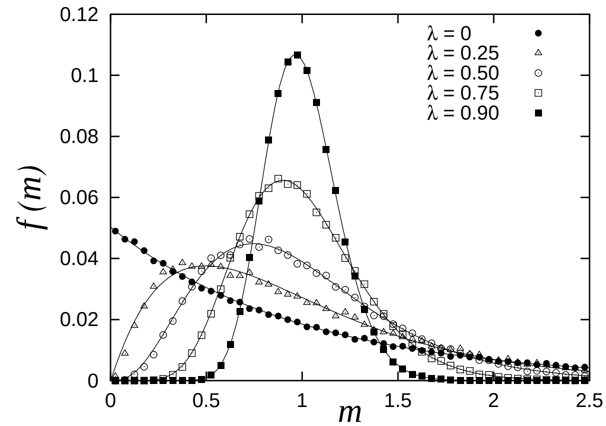

Fig. 1. Equilibrium money distributions for different values of the saving propensity λ, in the closed economy model defined by Eqs. (5). The continuous curves are the fitting functions, defined in Eq. (8).

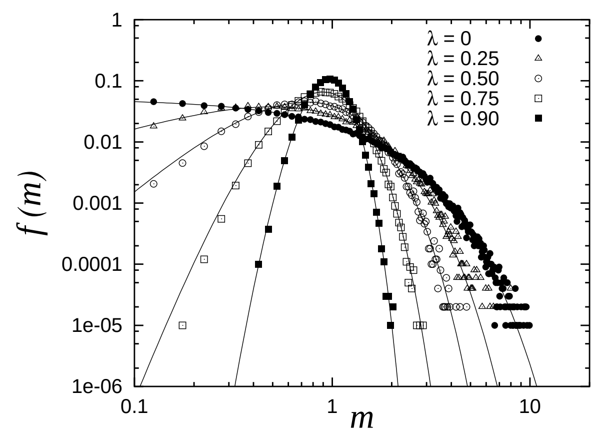

Fig. 2. As in Fig. (1), but on a double logarithmic scale.

the agent money in units of the average money $\langle m \rangle$, and the parameter

$$n(\lambda) = 1 + \frac{3\lambda}{1-\lambda} \; . \tag{7}$$

We found that the money distributions, for arbitrary values of λ, are well fitted by the function

$$P(x) = a_n x^{n-1} \exp\left(-nx\right) \; , \tag{8}$$

where x and n are defined in Eqs. (6) and (7), respectively [1]. Using normal-

[1] An excellent fitting is also obtained if the variable in the exponential is raised to a power c, $\exp(-nx) \to \exp(-nx^c)$, where c is an additional parameter and the value of c is close to one for all values of λ. Here we assume $c \equiv 1$, since the corresponding fitting is good.

ization conditions, the prefactor is easily shown to be given by

$$a_n = \frac{n^n}{\Gamma(n)} \ , \tag{9}$$

where $\Gamma(n)$ is the Gamma function.

The fitting curves for the distribution (continuous lines) are compared with the numerical data in Fig. 1. The fitting describes the distribution also at large values of x, as shown by the logarithmic plots in Fig. 2. The numerical values of the parameters a_n and n are compared with the respective fitting functions (9) and (7) in Fig. 3. The distribution function (8) still contains an exponential factor $\exp(-nx)$, similar to that of the Gibbs distribution, but the average value is now rescaled by n. The power x^{n-1} qualitatively changes the Gibbs distribution into a curve with a maximum at $x > 0$, i. e. with a mode different from zero.

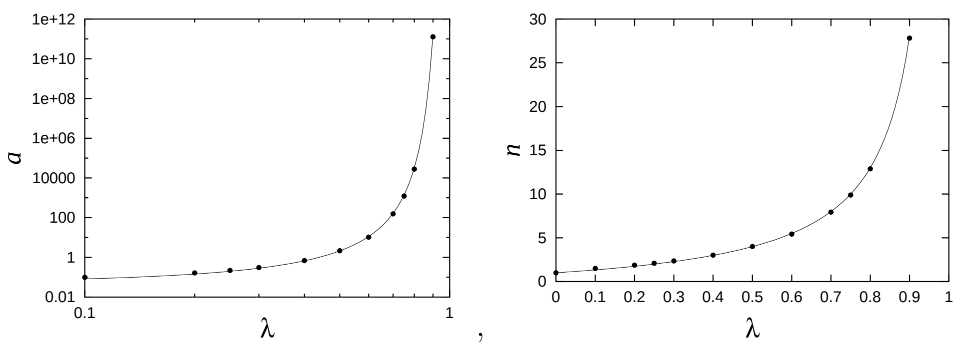

Fig. 3. The parameters a_n (left) and n (right) versus λ obtained from numerical data (dots) and the corresponding analytical formulas (continuous curves) given by Eqs. (9) and (7), respectively.

It is to be noticed that by introducing the rescaled variable

$$x_n = nx \equiv \frac{m}{\langle m \rangle / n} \ , \tag{10}$$

and the corresponding probability density $P_n(x_n) = dF(x)/dx_n \equiv P(x)/n$, where $F(x)$ is the cumulative function, and using the explicit expression of a_n, Eq. (9), the distribution (8) becomes

$$P_n(x_n) = \frac{1}{\Gamma(n-1)} \, x_n^{n-1} \exp\left(-x_n\right) \ , \tag{11}$$

which reduces to the Poisson distribution for integer values of n.

4 Conclusions and discussion

We have studied a generalization of the simple closed economy model, in which a random reassignment of the agent money takes place, by introducing a saving propensity $\lambda > 0$. We have empirically obtained the corresponding exact analytical solution from a fitting of the numerical data. The distribution naturally lends itself to be interpreted as a Poisson distribution $P(n, x_n)$ for the reduced variable $x_n = m/(\langle m \rangle /n)$. The parameter $n = 1+3\lambda/(1-\lambda)$ is in principle continuous but it can vary between 1 and ∞ when λ varies between 0 and 1. This result raises the problem of a more rigorous derivation of the solution as well as of a deeper physical interpretation of the result.

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